Network User Equilibrium with Elastic Demand: Formulation, Qualitative Analysis and Computation

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Abstract

In this paper we present a differential variational inequality formulation of dynamic network user equilibrium with elastic travel demand. We discuss its qualitative properties and provide algorithms for and examples of its solution.

Keywords: *dynamic user equilibrium; differential variational inequalities; differential algebraic equations; dual time scales; fixed-point algorithm in Hilbert space*

1 Introductory remarks

This paper presents a differential variational inequality representation an elastic demand extension of the fixed-demand dynamic traffic assignment model originally presented in Friesz et al. (1993) and discussed subsequently by Friesz and Mookherjee (2006); Friesz et al. (2011, 2012); Friesz and Meimand (2012). As such, it is concerned with a specific type of dynamic traffic assignment known as dynamic user equilibrium (DUE) for which travel cost, including delay as well as early and late arrival penalties, are equilibrated and demand is determined endogenously.

2 Some history

In this section we review some of the few prior efforts to model DUE with elastic travel demand. This review is based in part on Friesz and Meimand (2012). Most of the studies of DUE reported in the DTA literature are about dynamic user equilibrium with constant travel demand for each origin-destination pair. It is, of course, not generally true that travel demand is fixed, even for short time horizons. Arnott et al. (1993) and Yang and Huang (1997) directly consider elastic travel demand; their work possesses a limited relationship to the analysis presented in this paper, for their work is concerned with a simple bottleneck instead of a nontrivial network, which is our focus.

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Yang and Meng (1998) extend a simple bottleneck model to a general queuing network with known elastic demand functions for each origin-destination (OD) pair. Wie et al. (2002) study a version of dynamic user equilibrium with elastic demand, using a complementarity formulation that requires path delays to be expressible in closed form. Szeto and Lo (2004) study dynamic user equilibrium with elastic travel demand when network loading is based on the cell transmission model (CTM); their formulation is based on discrete time and is expressed as a finite-dimensional variational inequality (VI). Han et al. (2011) study dynamic user equilibrium with elastic travel demand for a network whose traffic flows are also described by CTM.

Although Friesz et al. (2011) show that analysis and computation of dynamic user equilibrium with constant travel demand is tremendously simplified by stating it as a a differential variational inequality (DVI), they do not discuss how elastic demand may be accommodated within a DVI framework. We know of no prior published work that extends the Friesz et al. (2011) formulation to an elastic demand setting. That is, our demonstration in this paper that dynamic user equilibrium with elastic travel demand may be formulated as a differential variational inequality in continuous time is original and has not been previously reported.

The DVI formulation of elastic demand DUE is not straightforward. In particular, the DVI presented herein has both infinite-dimensional and finite-dimensional terms; moreover, for any given origin-destination pair, inverse travel demand corresponding to a dynamic user equilibrium depends on the terminal value of a state variable representing cumulative departures.

The DVI formulation achieved in this paper is significant because it allows the still emergening theory of differential variational inequalities to be employed for the analysis and computation of solutions of the elastic-demand DUE problem when simultaneous departure time and route choice are within the purview of users, all of which constitutes a foundation problem within the field of dynamic traffic assignment. A good review of recent insights into abstract differential variational inequality theory, including computational methods for solving such problems, is provided by Pang and Stewart (2008). Also, differential variational inequalities involving the kind of explicit, agent-specific control variables employed herein are presented in Friesz (2010).

3 Notation and essential background

We assume the same general setup as Friesz and Meimand (2012). In particular, the time interval of analysis is a single commuting period or "day" expressed as

$$[t_0, t_f] \subset \Re^1_+$$

where $t_f > t_0$, and both t_0 and t_f are fixed. Here, as in all DUE modeling, the single most crucial ingredient is the path delay operator, which provides the delay on any path p per unit of flow departing from the origin of that path; it is denoted by

$$D_p(t, h) \qquad \text{for all } p \in \mathcal{P}$$

where $\mathcal{P}$ is the set of all paths employed by travelers, t denotes departure time, and h is a vector of departure rates. From these, we construct effective unit path delay operators $\Psi_p(t, h)$ by adding the so-called schedule delay $E\left[t + D_p(t, h) - T_A\right]$; that is

$$\Psi_p(t, h) = D_p(t, h) + E\left[t + D_p(t, h) - T_A\right] \qquad \text{for all } p \in \mathcal{P}$$

where T_A is the desired arrival time and $T_A < t_f$. The function $E(\cdot)$ assesses a penalty whenever

$$t + D_p(t,h) \neq T_A \tag{3.1}$$

since $t + D_p(t,h)$ is the clock time at which departing traffic arrives at the destination of path $p \in \mathcal{P}$. We stipulate that each

$$\Psi_p(\cdot,h) : [t_0,t_f] \longrightarrow \Re^1_{++} \quad \text{for all } p \in \mathcal{P}$$

is measurable and strictly positive. We employ the obvious notation

$$(\Psi_p(\cdot,h) : p \in \mathcal{P}) \in \Re^{|\mathcal{P}|}$$

to express the complete vector of effective delay operators.

It is now well known that path delay operators may be obtained from an embedded delay model, data combined with response surface methodology, or data combined with inverse modeling. Unfortunately, regardless of how derived, realistic path delay operators do not possess the desirable property of monotonicity; they may also be non-differentiable. However, we shall have more to say about path delays in a subsequent section of this paper.

Transportation demand is assumed to be expressed as the following invertible function

$$Q_{ij}(t_f) = F_{ij}[v]$$

for each origin-destination pair $(i,j) \in \mathcal{W}$, where $\mathcal{W}$ is the set of all origin-destination pairs and v is a concatenation of origin-destination minimum travel costs v_{ij} associated with $(i,j) \in \mathcal{W}$. That is, we have that

$$\begin{aligned} v_{ij} &\in \Re^1_+ \\ v &= (v_{ij} : (i,j) \in \mathcal{W}) \in \Re^{|\mathcal{W}|} \end{aligned}$$

Note that to say v_{ij} is a minimum travel cost means it is the minimum cost for all departure time choices and all route choices pertinent to origin-destination pair $(i,j) \in \mathcal{W}$. Further note that $Q_{ij}(t_f)$ is the unknown cummulative travel demand between $(i,j) \in \mathcal{W}$ that must ultimately arrive by time t_f.

We will also find it convenient to form the complete vector of travel demands by concatenating the origin-specific travel demands to obtain

$$\begin{aligned} Q &= (Q_{ij}(t_f) : (i,j) \in \mathcal{W}) \in \Re^{|\mathcal{W}|} \\ F &: \Re^{|\mathcal{W}|}_+ \longrightarrow \Re^{|\mathcal{W}|}_+ \end{aligned}$$

The inverse demand function for every $(i,j) \in \mathcal{W}$ is

$$v_{ij} = \Theta_{ij}[Q(t_f)],$$

and we naturally define

$$\Theta = (\Theta_{ij} : (i,j) \in \mathcal{W}) \in \Re^{|\mathcal{W}|}$$

Additionally, we will define the set $\mathcal{P}_{ij}$ to be the set of paths that connect origin-destination pair $(i,j) \in \mathcal{W}$. We denote the space of square integrable functions for the real interval $[t_0,t_f]$ by $L^2[t_0,t_f]$. We stipulate that

$$h \in \left(L^2_+[t_0,t_f]\right)^{|\mathcal{P}|}$$

We write the flow conservation constraints as

$$\sum_{p\in\mathcal{P}_{ij}} \int_{t_0}^{t_f} h_p(t)\,dt = Q_{ij}(t_f) \qquad \text{for all } (i,j)\in\mathcal{W} \tag{3.2}$$

where (3.2) is comprised of Lebesgue integrals. In (3.2), we consider demand to be the cumulative travel that takes place by the end of the analysis horizon t_f. We define the set of feasible flows for each $v \in \Re^{|\mathcal{W}|}$ by

$$\Lambda_0 = \left\{ h \geq 0 : \sum_{p\in\mathcal{P}_{ij}} \int_{t_0}^{t_f} h_p(t)\,dt = Q_{ij}(t_f) \qquad \text{for all } (i,j)\in\mathcal{W} \right\} \subseteq \left(L_+^2[t_0,t_f]\right)^{|\mathcal{P}|}$$

Given h let us also define the *essential infimum* of effective travel delays

$$v_{ij} = ess\inf\left[\Psi_p(t,h) : p\in\mathcal{P}_{ij}\right] \qquad \text{for all } (i,j)\in\mathcal{W}$$

We rely on the following definition of dynamic user equilibrium:

Definition 3.1. *Dynamic user equilibrium. A vector of departure rates (path flows) $h^* \in \Lambda_0$ is a dynamic user equilibrium if*

$$h_p^*(t) > 0, p\in P_{ij} \Longrightarrow \Psi_p[t, h^*(t)] = v_{ij}$$

We denote this equilibrium by $DUE(\Psi, \Lambda_0, [t_0, t_f])$.

The meaning of Definition 3.1 is clear: positive departure rates at a particular time along a particular path must coincide with least effective travel delay. Furthermore

$$\Psi_p(t,h^*) > v_{ij}, p\in P_{ij} \Longrightarrow h_p^* = 0$$

is an implication of Definition 3.1.

4 Dynamics

We assume there are unknown terminal state variables $Q_{ij}(t_f)$, for all $(i,j)\in\mathcal{W}$, which are the realized DUE travel demands. Moreover, for each origin-destination pair $(i,j)\in\mathcal{W}$, inverse travel demand is expressed as

$$v_{ij} = \Theta_{ij}[Q(t_f)] \tag{4.3}$$

Thus, the $Q_{ij}(t_f)$, for all $(i,j)\in\mathcal{W}$, will be determined endogenously to the differential variational inequality presented subsequently in Section 5. Such an approach contrasts to the approach employed by Friesz et al. (2010) to study fixed-demand DUE by making each $Q_{ij}(t_f)$ an a priori fixed constant. Accordingly, we introduce the following dynamics:

$$\frac{dQ_{ij}}{dt} = \sum_{p\in\mathcal{P}_{ij}} h_p \qquad Q_{ij}(t_0) = 0 \qquad \text{for all } (i,j)\in\mathcal{W} \tag{4.4}$$

As a consequence, we employ the following alternative form of the feasible set:

$$\Lambda = \left\{ h\geq 0 : \frac{dQ_{ij}}{dt} = \sum_{p\in P_{ij}} h_p(t) \qquad Q_{ij}(t_0) = 0 \quad \text{for all } (i,j)\in\mathcal{W} \right\} \subseteq \left(L_+^2[t_0,t_f]\right)^{|\mathcal{P}|} \tag{4.5}$$

Note that the feasible set Λ in (4.5) is expressed as a set of path flows since knowledge of h completely determines the demands that satisfy the inital value problem (4.4).

5 The Differential Variational Inequality

Experience with differential games in continuous time suggests that an elastic demand dynamic user equilibrium is equivalent to the following variational inequality under suitable regularity conditions: find $h^* \in \Lambda$ such that

$$\sum_{p\in\mathcal{P}} \int \int_{t_0}^{t_f} \Psi_p(t,h^*)(h_p - h_p^*)dt - \sum_{(i,j)\in\mathcal{W}} \Theta_{ij}^* \left[Q^*\left(t_f\right)\right] \left[Q_{ij}\left(t_f\right) - Q_{ij}^*\left(t_f\right)\right] \geq 0 \quad \text{for all } h \in \Lambda \tag{5.6}$$

where $Q^*(t_f)$ is the terminal equilibrium demand and $\Theta^*\left[Q^*\left(t_f\right)\right]$ is the inverse demand function evaluated at $Q^*(t_f)$. We refer to differential variational inequality (5.6) as $DVI(\Psi, \Theta, t_0, t_f)$. We now proceed constructively to show that (5.6) is equivalent to elastic demand DUE for appropriate regularity conditions.

In particular we note that the differential variational variational inequality of interest may be written as

$$\begin{aligned}\sum_{(i,j)\in\mathcal{W}} \sum_{p\in\mathcal{P}_{ij}} \int_{t_0}^{t_f} \Psi_p(t,h^*)h_p dt - \sum_{(i,j)\in\mathcal{W}} \Theta_{ij}^*\left[Q^*\left(t_f\right)\right] Q_{ij}\left(t_f\right) \\ \geq \sum_{(i,j)\in\mathcal{W}} \sum_{p\in\mathcal{P}_{ij}} \int_{t_0}^{t_f} \Psi_p(t,h^*)h_p^* dt - \sum_{(i,j)\in\mathcal{W}} \Theta_{ij}^*\left[Q^*\left(t_f\right)\right] Q_{ij}^*(t_f)\end{aligned} \tag{5.7}$$

for all $h \in \Lambda$. Inequality (5.7) means that the solution $h^* \in \Lambda$ satisfies the optimal control problem

$$\min J_0 = -\sum_{(i,j)\in\mathcal{W}} \Theta_{ij}^*\left[Q^*\left(t_f\right)\right] Q_{ij}\left(t_f\right) + \sum_{(i,j)\in\mathcal{W}} \sum_{p\in\mathcal{P}_{ij}} \int_{t_0}^{t_f} \Psi_p(t,h^*)h_p dt \tag{5.8}$$

subject to

$$\frac{dQ_{ij}}{dt} = \sum_{p\in\mathcal{P}_{ij}} h_p\left(t\right) \quad \text{for all } (i,j) \in \mathcal{W} \tag{5.9}$$

$$Q_{ij}(t_0) = 0 \quad \text{for all } (i,j) \in \mathcal{W} \tag{5.10}$$

$$h \geq 0 \tag{5.11}$$

This optimal control problem may not be used for computation, since it involves knowledge of the DVI solution. However, it may be used to express necessary and sufficient conditions for the solution of $DVI(\Psi, \Theta, t_0, t_f)$. In particular, the Hamiltonian for $DVI(\Psi, \Theta, t_0, t_f)$ is

$$\begin{aligned} H &= \sum_{(i,j)\in\mathcal{W}} \sum_{p\in\mathcal{P}_{ij}} \Psi_p(t,h^*)h_p + \sum_{(i,j)\in\mathcal{W}} \lambda_{ij} \sum_{p\in\mathcal{P}_{ij}} h_p \\ &= \sum_{(i,j)\in\mathcal{W}} \left\{ \sum_{p\in\mathcal{P}_{ij}} \left[\Psi_p(t,h^*) + \lambda_{ij}\right] h_p \right\} \end{aligned}$$

where the adjoint equations are

$$\frac{d\lambda_{ij}}{dt} = -\frac{\partial H}{\partial Q_{ij}} = 0 \quad \text{for all } (i,j) \in \mathcal{W}, p \in \mathcal{P}_{ij}, t \in [t_0, t_f] \tag{5.12}$$

with transversality conditions

$$\lambda_{ij}(t_f) = -\frac{\partial \sum_{(i,j)\in\mathcal{W}} \Theta^*_{ij}[Q^*(t_f)]\, Q_{ij}(t_f)}{\partial Q_{ij}(t_f)} = -\Theta^*_{ij}[Q^*(t_f)] \quad \text{for all } (i,j)\in\mathcal{W}, p\in\mathcal{P}_{ij}, t\in[t_0,t_f] \tag{5.13}$$

It is clear from (5.12) and (5.13) that

$$\lambda_{ij}(t) = -\Theta^*_{ij}[Q^*(t_f)]\,,\text{ a constant}$$

We note that the Hamiltonian is linear in h and does not depend explicitly on the state variables. By Theorem 3.7 of Friesz (2010), the Mangasarian sufficiency theorem assures the minimum principle and associated necessary conditions are also sufficient.

Since h is a control vector and must obey the minimum principle in $\Re^{|\mathcal{P}|}$ for each instant of time, we enforce

$$\min_h H \quad \text{s.t.} \quad -h \le 0$$

for which the Kuhn-Tucker conditions are

$$\Psi_p(t,h^*) - \Theta^*_{ij}[Q^*(t_f)] = \rho_p \ge 0 \quad \text{for all } (i,j)\in\mathcal{W}, p\in\mathcal{P}_{ij}, t\in[t_0,t_f] \tag{5.14}$$

where the ρ_p are dual variables satisfying the complementary slackness conditions

$$\rho_p h_p = 0 \quad \text{for all } (i,j)\in\mathcal{W}, p\in\mathcal{P}_{ij}, t\in[t_0,t_f] \tag{5.15}$$

From (5.14) and (5.15) we have immediately

$$\begin{aligned} h^*_p &> 0, p\in\mathcal{P}_{ij} \Longrightarrow \Psi_p(t,h^*) = \Theta^*_{ij}[Q^*(t_f)] \\ \Psi_p(t,h^*) &> \Theta^*_{ij}[Q^*(t_f)]\,, p\in\mathcal{P}_{ij} \Longrightarrow h^*_p = 0 \end{aligned}$$

which are recognized as conditions describing a dynamic user equilibrium.

We have noted above that (5.14) is equivalent to a dynamic user equilibrium; thus, showing that (5.14) corresponds to a solution of $DVI(\Psi,\Theta,t_0,t_f)$ will complete the demonstration that $DVI(\Psi,\Theta,t_0,t_f)$ is equivalent to a dynamic user equilibrium. In particular note that

$$\rho_p(h_p - h^*_p) = \rho_p h_p - \rho_p h^*_p = \rho_p h_p \ge 0$$

so that

$$\left\{\Psi_p(t,h^*) - \Theta^*_{ij}[Q^*(t_f)]\right\}(h_p - h^*_p) \ge 0 \quad \text{for all } (i,j)\in\mathcal{W}, p\in\mathcal{P}_{ij}, t\in[t_0,t_f] \tag{5.16}$$

where $h, h^* \in \Lambda$. From (5.16) we obtain

$$\Psi_p(t,h^*)(h_p - h^*_p) - \Theta^*_{ij}[Q^*(t_f)](h_p - h^*_p) \ge 0 \quad \text{for all } (i,j)\in\mathcal{W}, p\in\mathcal{P}_{ij}, t\in[t_0,t_f] \tag{5.17}$$

From (5.17), we obtain

$$\sum_{p\in\mathcal{P}} \int\int_{t_0}^{t_f} \Psi_p(t,h^*)(h_p - h^*_p)dt - \sum_{(i,j)\in\mathcal{W}} \Theta^*_{ij}[Q^*(t_f)] \int\int_{t_0}^{t_f} (\sum_{p\in\mathcal{P}_{ij}} h_p - \sum_{p\in\mathcal{P}_{ij}} h^*_p)dt \ge 0 \tag{5.18}$$

Therefore

$$\sum_{p\in\mathcal{P}}\int\int_{t_0}^{t_f}\Psi_p(t,h^*)(h_p-h_p^*)dt-\sum_{(i,j)\in\mathcal{W}}\Theta_{ij}^*\left[Q^*\left(t_f\right)\right]\int\int_{t_0}^{t_f}(\frac{dQ_{ij}}{dt}-\frac{dQ_{ij}^*}{dt})dt\geq 0, \quad (5.19)$$

from which $DVI(\Psi,\Theta,t_0,t_f)$ is obtained immediately, since $Q_{ij}(t_0)=Q_{ij}^*(t_0)=0$.

The preceding analysis has established the validity of the following theorem:

Theorem 5.1. *Elastic demand dynamic user equilibrium is equivalent to a differential variational inequality. Assume $\Psi_p(\cdot,h):[t_0,t_f]\longrightarrow\Re_{++}^1$ is measurable and strictly positive for all $p\in\mathcal{P}$ and all $h\in\Lambda$. Also assume that the elastic travel demand function is invertible, with inverse $\Theta_{ij}\left(Q\right)$ for all $(i,j)\in\mathcal{W}$. A vector of departure rates (path flows) $h^*\in\Lambda$ is a dynamic user equilibrium with associated demand $Q^*(t_f)$ if and only if h^* solves $DVI(\Psi,\Theta,t_0,t_f)$.*

6 Additional qualitative properties

We are presently working on existence results, which will be ready by the time of submission of the full paper. Those results will be wholly original. Also existence will be discussed from the point of view of the path delay operators that are assumed or derived from network loading considerations. We will conduct analysis on several network flow models including the Vickrey model (Vickrey, 1969; Han et al., 2012a,b), the LWR-Lax model (Friesz et al., 2012), the Link Delay Model Friesz et al. (1993); Han et al. (2012d), and the kinematic wave model (Han et al., 2012e). We have analyzed the continuity of the effective delay operators in Hilbert space. In particular, such an analytical property has been established for the Vickrey model (Han et al., 2012c) and the link delay model Han et al. (2012d). We have also acquired the capability of analytically investigating the kinematic wave model on networks with vehicle spillback. In particular, existence, uniqueness as well as well-posedness of a network-based kinematic wave model has been established in Han et al. (2012e).

These qualitative results on network loading models, when combined with the general framework of proving DUE existence proposed by Han et al. (2012c), will easily lead to the existence results of DUE with different network performance models.

7 Computation

The computation of solutions to (5.6) is also under development and will be ready by the time of submission of the full paper. The algorithms and calculations will be wholly original. We have successfully created analytical DUE models supported by an efficient DUE solution algorithm that has been extensively tested, see Friesz et al. (2012, 2011). We expect to finish and demonstrate a DUE solution associated with the kinematic wave model with spillback for large-scale network by the submission of the full paper.

8 Conclusion

We have shown dynamic network user equilibrium based on simultaneous departure time and route choice in the presence of elastic travel demand may be formulated as a differential variational inequality (DVI). The above conclusions will be expanded to reflect the qualitative and numerical analyses of Sections 6 and 7.

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